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GENERALIZED MANDELBROT RULE FOR FRACTAL SECTIONS

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Mandelbrot's rule for sections is generalized to apply to the Hentschel and Procaccia fractal dimension at arbitrary q and on arbitrary sections. It is shown that for almost all (n-m)-dimensional sections, $D^n(q) = D^{n-m}(q) + m$, where the $D^r(q)$ are box-counting Hentschel and Procaccia generalized fractal dimensions of r-dimensional sections of homogeneous fractal point sets in R^n. The rule is shown to apply for finite "thickness" sections as well as "true" sections. A more general form of the rule applicable to inhomogeneous fractal sets is also presented.

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INTRODUCTION

Mandelbrot's rule (ref 1) states that for almost all (n-1)-dimensional sections

$$D_{HB}^{n} = D_{HB}^{n-1} + 1$$

where the D_{HB}^{r} are Hausdorff-Bouligand dimensions of r-dimensional sections of homogeneous fractal sets in R^n. This important rule comprises the basis for a substantial body of numerical fractal analysis.

The following generalization of Mandelbrot's rule is demonstrated here: For almost all (n-m)-dimensional sections

$$D^{n}(q) = D^{n-m}(q) + m \tag{1}$$

where the $D^r(q)$ are box-counting Hentschel and Procaccia (refs 2,3) generalized fractal dimensions of r-dimensional sections of homogeneous fractal point sets in R^n. Equation (1) obtains for arbitrary q and $n > m \geq 1$. The $q = 0$, $m = 1$ form of Eq. (1) is the box-counting formulation of the standard Mandelbrot rule.

THEORY

Let the point set in R^n be represented as points in an arbitrarily-oriented Cartesian coordinate system. Define an ensemble of (n-m)-dimensional "sections,"

$$T_{n,m}(J) \equiv T_{n,m}(\{X_i(J) \mid i = 1,m\};\ \delta)$$

$$\equiv \{\vec{x} \mid X_i(J) < x_i \leq X_i(J) + \delta\ ,\ i = 1,m;\ -L < x_j \leq L;\ j = m+1,n\} \tag{2}$$

where L is large enough to cover a substantial portion of the fractal set, δ is "small," and the $\{X_i(J)\}$ are such that the $T_{n,m}(J)$ fill a region of R^n without overlaps or gaps. We refer to $T_{n,m}(J)$ sections as "J-strips."

Define sets of boxes (hypercubes) of side E aligned along the coordinate axes in R^n. Let each hypercube in R^n be identified by the position of one of its corners in units of E, i.e., the box at (I(1),...,I(n))E is indexed [I(1),...,I(n)]. Such sets of boxes define corresponding sets of boxes of side E and "thickness" δ^m in each J-strip; the box corresponding to the [I(1),...,I(n)] box, which intersects the J-strip, induces the box, which is indexed [J;I(m+1),...,I(n)], in the J-strip. There will be approximately E/δ J-strips, which intersect a given box in R^n. Figure 1 illustrates the definitions in R^2.

Box-counting fractal analysis is generally applied to large, but finite, subsets of fractal sets. The fractal subsets are assumed to be large enough to yield converged box-counting dimensions. Small δ implies that $\delta << E_{min}$, where E_{min} is the smallest hypercube edge. The present results also apply for complete fractal sets and in the limit $\delta \to 0$.

Let $p_{I(1),...,I(n)}$ denote the occupation probability for the [I(1),...,I(n)] box of side E in R^n, which is defined as the number of members of the subset of the fractal point set in the [I(1),...,I(n)] box divided by the number of members of the subset of the fractal point set in all the boxes in R^n.

Let $N_{J;I(m+1),...,I(n)}$ be the number of points of the fractal subset in the [J;I(m+1),...,I(n)] box.

Let S_J be the number of members of the subset of the fractal point set in the J-strip:

$$S_J = \sum_{I(m+1),\dots,I(n)} N_{J;I(m+1),\dots,I(n)} .$$

S_J is proportional to the J-strip probability σ_J in R^n, where

$$\sigma_J \equiv S_J/N$$

$$and\ N = \sum_J S_J = \sum_{J,I(m+1),\dots,I(n)} N_{J;I(m+1),\dots,I(n)}$$

is the total number of members in the fractal subset. The occupation probability for the $[J;I(m+1),\dots,I(n)]$ box is $\pi_{J;I(m+1),\dots,I(n)} \equiv N_{J;I(m+1),\dots,I(n)}/S_J$. By definition

$$\sum_{I(m+1),\dots,I(n)} \pi_{J;I(m+1),\dots,I(n)} = 1 .$$

The partition function $Z^n(q,E)$ in R^n is defined as

$$Z^n(q,E) \equiv \sum_{I(1),\dots,I(n)} p^q_{I(1)\dots I(n)} . \tag{3a}$$

Since the J-strip box corresponding to the $[I(1),\dots,I(n)]$ box in R^n has volume $\Omega_J = E^n(\delta/E)^m$,

$$p_{I(1),\dots,I(n)} \approx (E/\delta)^m \sigma_J \pi_{J;I(m+1),\dots,I(n)} . \tag{3b}$$

Furthermore, each box in R^n is counted approximately $(E/\delta)^m$ times in the sum over strips. Thus, the partition function $Z^n(q,E)$ in R^n can be expanded as

$$Z^n(q,E) = \sum_J \sum_{I(m+1)\dots I(n)} \left[\frac{\delta}{E}\right]^m \left(\left[\frac{E}{\delta}\right]^m \pi_{J;I(m+1),\dots,I(n)} \sigma_J\right)^q$$

$$= \left[\frac{E}{\delta}\right]^{(q-1)m} \sum_J \sigma_J^q \, Z^{n-m}(J;q,E) . \tag{3c}$$

where $Z^{n-m}(J;q,E)$ is the J-strip partition function. The approximations become exact in the limit $E \to 0$ with $E/\delta >> 1$ and are "good" approximations in the spirit of box-counting algorithms for "small enough" E with $E/\delta >> 1$; exact equalities obtain in the $E \to 0$ limit for true sections also.

Box-counting generalized fractal dimensions are defined as

$$D^n(q) = \left\{\frac{\partial \ln(Z^n(q,E))}{\partial \ln(E)}\right\}_{E_0} \tag{4a}$$

$$D^{n-m}(J;q) = \left\{\frac{\partial \ln(Z^{n-m}(J;q,E))}{\partial \ln(E)}\right\}_{E_0} \tag{4b}$$

where the right-hand sides are usually determined by least squares fitting over a range of "small" yardstick values centered about a yardstick E_0.

Since $\ln(Z^n(q,E))$ and $\ln(Z^{n-m}(J;q,E))$ scale like $(q-1)D^n(q)\ln(E)$ and $(q-1)D^{n-m}(J;q)\ln(E)$, respectively, Eq. (4) implies that

$$D^n(q) = \langle D^{n-m}(q)\rangle_{E_0} + m \tag{5}$$

where

$$\langle D^r(q)\rangle_{E_0} = \sum_J D^r(J;q)Z^r(J;q,E_0)\sigma_J^q \Big/ \sum_{J'} Z^r(J';q,E_0)\sigma_{J'}^q . \tag{6}$$

(Since the strip partition functions $Z^{n-m}(J;q,E_0)$ are of the same order of magnitude for given q and E_0,

$$\langle D^r(q)\rangle_{E_0} \approx \sum_J D^r(J;q)\sigma_J^q \Big/ \sum_{J'} \sigma_{J'}^q \tag{6'}$$

might be a useful approximation.)

N.B., Eqs. (5), (6), and (6') may be interpreted for inhomogeneous fractal sets.

Excluding "exceptional sections," the $D^r(J;q)$ are independent of J for homogeneous fractal point sets and Eq. (5) is equivalent to the generalized form of Mandelbrot's rule for box-counting fractal dimensions at arbitrary q and almost all (n-m)-dimensional J-strips

$$D^n(q) = D^{n-m}(J;q) + m \text{ , for almost all } J . \tag{7}$$

The existence of "exceptional sections" related to special symmetries in fractal point sets, as discussed by Mandelbrot (ref 1), necessitates the inclusion of the "almost all" caveat in the statement of the rule for sections.

The steps in the derivation of Eqs. (5) through (7) hold in the $\delta \to 0$ limit. Thus, the principal results hold not only for finite "thickness" (finite δ) sections, but they also hold for "true" sections of fractal sets.

SUMMARY AND DISCUSSION

Mandelbrot's rule relates the fractal dimension (for $q = 0$) of a fractal point set in R^n, $D^n(0)$, to the fractal dimension (for $q = 0$) of subsets of the fractal point set in (n-1)-dimensional sections, $D^{n-1}(0)$. The principal results of the present work, contained in Eqs. (5) through (7), generalize Mandelbrot's rule for sections to apply for box-counting Hentschel and Procaccia fractal dimensions $D^r(q)$ at arbitrary q for r-dimensional sections of small "thickness" δ with $n \geq r > 0$. Small δ implies that $\delta << E_0$, where E_0 is defined after Eq. (4). Furthermore, the equations hold in the $\delta \rightarrow 0$ limit, and thus apply for "true" sections.

One might work with finite δ sections to analyze point sets generated by nonlinear mappings or the distribution of galaxies. One might work with "true" two-dimensional sections of three-dimensional surfaces in a multifractal generalization of the slit-island method (ref 4) (which determines D(0) of fracture surfaces by perimeter-area analysis of fracture surface sections).

Since sparse regions of a fractal subset dominate the average in Eq. (6) for $q < 0$, numerical problems in box-counting evaluations of $D^n(q)$ are likely at $q < 0$. Modified box-counting strategies, which put less stress on sparsely occupied strips, might improve fractal analysis at negative q.

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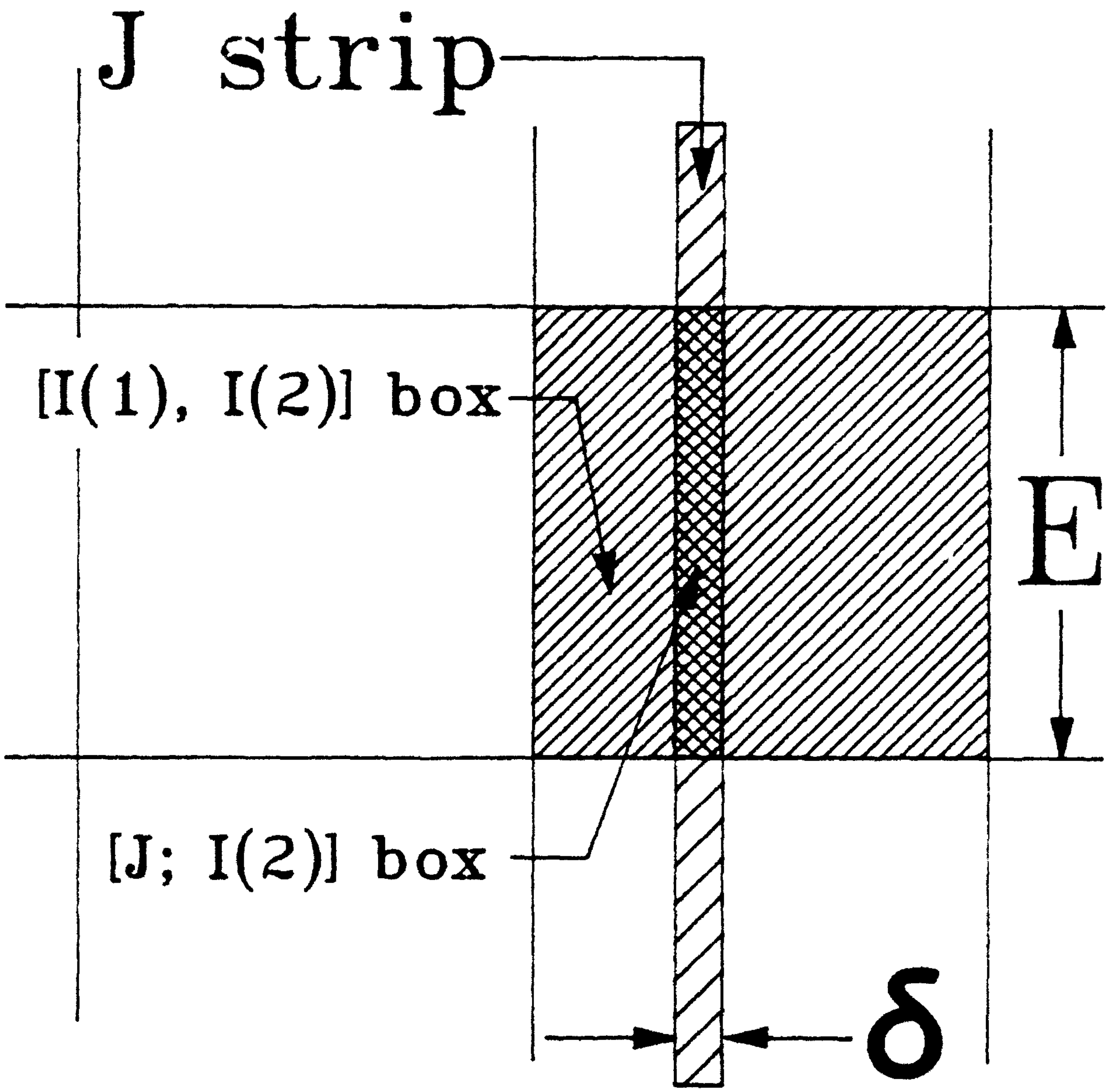

Figure 1. Illustration of the box and strip definitions in R^2. The box indexed [I(1),I(2)] of side E, whose "volume" is E^2 in R^2, induces the box indexed [J;I(2)] of side E in the J-strip. The J-strip is a one-dimensional section of "thickness" δ.

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